The Sum of the First n Whole Number Cubes

Problem:

a) Derive the formula for the sum of the first n cubes.

b) Prove the sum is the square of the n^{th} triangular number.

$$\left(\text{i. e.} S_3 = 1^3 + 2^3 + 3^3 + \ \ldots\ldots\ldots + n^3 = (S_1)^2 = \left(\frac{n(n+1)}{2}\right)^2 = \frac{n^2(n+1)^2}{4}\right)$$

Solution:

a) Let's examine the first five sums:

$$1^3 = 1$$
$$1^3 + 2^3 = 9 = 3^2 = (1+2)^3$$
$$1^3 + 2^3 + 3^3 = 36 = 6^2 = (3+3)^2$$
$$1^3 + 2^3 + 3^3 + 4^3 = 100 = 10^2 = (6+4)^2$$
$$1^3 + 2^3 + 3^3 + 4^3 + 5^3 = 225 = 15^2 = (10+5)^2$$

So it seems that the sum is always square, more specifically the sum of first n cubes

$$1^3 + 2^3 + 3^3 + \ \ldots\ldots\ldots\ldots\ldots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$$

We can verify the formula, $1^3 + 2^3 + 3^3 + \ \ldots\ldots\ldots\ldots\ldots + 10^3 = \left(\frac{10\cdot 11}{2}\right)^2 = 55^2 = 3025$

b) Now, let's prove the formula.

$$\sum_{k=1}^{n} k^4 - (k-1)^4$$

$$= [n^4 - (n-1)^4] + [(n-1)^4 - (n-2)^4] + \ \ldots\ldots\ldots\ldots + \&[3^4 - 3^4]$$
$$+ [2^4 - 1^4] + [1^4 - 0^4]$$

$$= n^4 \ \ldots\ldots\ldots\ldots\ldots (i)$$

Now, $k^4 - (k-1)^4 = k^4 - (k^4 - 4k^3 + 6k^2 - 4k + 1) = 4k^3 - 6k^2 + 4k - 1 \ldots\ldots\ldots (ii)$

Hence,

$$\sum_{k=1}^{n}(4k^3 - 4k^2 + 4k - 1) = 4\sum k^3 - 6\sum k^2 + 4\sum k - \sum 1$$

$$= 4\sum k^3 - 6\frac{n(n+1)(2n+1)}{6} + 4\frac{n(n+1)}{2} - n$$

$$= 4\sum k^3 - n(n+1)(2n+1) + 2n(n+1) - n \dots\dots\dots (iii)$$

Now using (i) and (ii), for equation (iii) we can rewrite

$$4\sum k^3 - n(n+1)(2n+1) + 2n(n+1) - n = n^4$$

$$\Rightarrow 4\sum k^3 = n^4 + n(n+1)(2n+1) - 2n(n+1) + n$$

$$= n[n^3 + (n+1)(2n+1) - 2(n+1) + 1]$$

$$= n[n^3 + 2n^2 + 3n + 1 - 2n - 2 + 1]$$

$$= n[n^3 + 2n^2 + n]$$

$$= n^2(n^2 + 2n + 1)$$

$$= n^2(n+1)^2$$

Hence,

$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4} = \left(\frac{n(n+1)}{2}\right)^2$$

So, the sum of first n cubes is the square of the sum of the first n natural numbers.